

DEGENERATE PRINCIPAL SERIES REPRESENTATIONS OF $\mathrm{SO}(p+1, p)$

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ABSTRACT. For p odd, the Lie group $G^\sharp = \mathrm{SO}_0(p+1, p+1)$ has a family of unitary degenerate principal series representations realized on the space of real $(p+1) \times (p+1)$ skew symmetric matrices, similar to the Stein's complementary series for $\mathrm{SL}(2n, \mathbb{C})$ or Speh's representation for $\mathrm{SL}(2n, \mathbb{R})$. We consider their restriction on the subgroup $G_0 = \mathrm{SO}_0(p+1, p)$ and prove that they are still irreducible and is equivalent to (a unitarization of) the principal series representation of G , and also irreducible under a maximal parabolic subgroup of G .

1. INTRODUCTION

In the present paper we shall study the unitarity of degenerate principal series representations of the group $G = \mathrm{SO}(p+1, p)$ induced from certain maximal parabolic subgroup for odd $p = 2q - 1$.

In the case of the group $\mathrm{SO}_0(n, n)$, or more generally $\mathrm{SU}(n, n; \mathbb{F})$, for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, Johnson [15] has determined the range of unitarity of the representations; for $\mathrm{SU}(n, n, \mathbb{F})$, n even, he found certain complementary series. Some generalizations of these results were obtained in [20, 19, 28, 12] for a larger class of groups by using computations in the compact picture. The analysis of these representations in the non-compact picture has been done in [2].

We shall prove that the restriction of the complementary series of $G^\sharp = \mathrm{SO}_0(p+1, p+1)$ to the opposite maximal parabolic subgroup of G is irreducible, and in particular the restriction to the identity component G_0 of G is irreducible. We shall use mostly the non-compact realization of the principal series. The proof relies on both Euclidean and nilpotent Fourier transform.

The restriction of the degenerate principal series representations of $\mathrm{SO}(n, n)$ to the subgroups $\mathrm{SO}(n, m) \times \mathrm{SO}(n-m)$ for $m < n$ has been studied earlier by Lee-Loke [17] in the compact picture, the representations of $\mathrm{SO}(n, m) \times \mathrm{SO}(n-m)$ appearing are of the form $\tau \times \tau'$, and the representations τ are degenerate principal series. It might be true that the representations τ of $\mathrm{SO}(n, m)$ are also irreducible under corresponding the maximal parabolic subgroup, as we show here for $m = n - 1$. We mention also that there has been quite some study of complementary series representations for semisimple Lie groups. In [1] a large class of complementary series representations is constructed with parabolic subgroups being cuspidal and maximal; our case of $\mathrm{SO}(p+1, p)$ here is however not cuspidal. For the groups

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$\mathrm{SO}(p, q)$ some complementary series similar to ours can be constructed by using the branching of holomorphic representations of $\mathrm{SU}(p, q)$ to $\mathrm{SO}(p, q)$. However those constructions works only for $q - p > 2$ (see [18, Theorem 3.1], [29, Theorem 5.2]), and thus they do not cover our present case of $\mathrm{SO}(p + 1, p)$. Our result about the irreducibility of the restriction to $\mathrm{SO}(p + 1, p)$ of representations of $\mathrm{SO}_0(p + 1, p + 1)$ is in a way similar to the Kirillov conjecture, now a theorem [3, 21] for $\mathrm{GL}_{p+1}(\mathbb{R})$ and $\mathrm{GL}_p(\mathbb{R}) \times \mathbb{R}^p$. See also [22] on the study of the restriction of complementary series of $\mathrm{SO}(n, 1)$ to $\mathrm{SO}(n - 1, 1)$, [13] on branching of highest weight representations, and [14] the classification of finitely decomposable representations of G^\sharp under G . We note also that the irreducibility result under the the parabolic group P can possibly be also proved abstractly by using the Mackey theory on induced representations. However we present a relatively elementary and direct proof, in particular it also yields a decomposition of the representation under the subgroup $\bar{N}\mathrm{Sp}(q - 1, \mathbb{R})$.

2. PRELIMINARIES

2.1. The group $G^\sharp = \mathrm{SO}_0(p + 1, p + 1)$ and $G = \mathrm{SO}(p + 1, p)$.

Let $M_{p,q}$ be the space of real $p \times q$ -matrices, and denote $M_p = M_{p,p}$. Denote $\mathcal{X} = \mathcal{X}_p = \{X \in M_p; X = -X^t\}$ the subspace of skew-symmetric real matrices. We will also use the short-hand notation $X^{-t} = (X^t)^{-1}$, for an invertible X . Denote I_n the identity matrix in M_n and $I_{p,q} = \mathrm{diag}(-I_p, I_q)$.

Let $p > 1$ and $G^\sharp = \mathrm{SO}_0(p + 1, p + 1)$ be the identity component of $\mathrm{SO}(p + 1, p + 1) = \{g \in M_{2p+2}; \det g = 1, g^t I_{p+1,p+1} g = I_{p+1,p+1}\}$, and $G = \mathrm{SO}(p + 1, p)$ realized as the subgroup of G^\sharp via:

$$G = \{\mathrm{diag}(g, 1) \in G^\sharp\} \subset G^\sharp.$$

The group G has two connected components and we denote its identity component by G_0 .

Elements g of G^\sharp will be written as 2×2 block matrices

$$g = \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix},$$

with each entry being in M_{p+1} . The Lie algebra \mathfrak{g}^\sharp of G^\sharp has the decomposition $\mathfrak{g}^\sharp = \mathfrak{k}^\sharp \oplus \mathfrak{p}^\sharp$ with $\mathfrak{k}^\sharp = \mathfrak{so}(p + 1) \oplus \mathfrak{so}(p + 1)$ with respect to the Cartan involution $g \rightarrow g^{-t}$. The group

$$K^\sharp = \{\mathrm{diag}(k_1, k_2); k_1, k_2 \in \mathrm{SO}(p + 1)\} = \mathrm{SO}(p + 1) \times \mathrm{SO}(p + 1),$$

is a maximal compact subgroup of G^\sharp with Lie algebra \mathfrak{k}^\sharp . Correspondingly $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{k} = \mathfrak{so}(p + 1) \oplus \mathfrak{so}(p)$ and

$$K_0 = \{\mathrm{diag}(k_1, k_2, 1); k_1 \in \mathrm{SO}(p + 1), k_2 \in \mathrm{SO}(p)\} \sim \mathrm{SO}(p + 1) \times \mathrm{SO}(p)$$

is a maximal compact subgroup of G_0 , while

$$K = \{\mathrm{diag}(k_1, k_2, \det k_2); k_1 \in \mathrm{SO}(p + 1), k_2 \in \mathrm{O}(p)\} \sim \mathrm{SO}(p + 1) \times \mathrm{O}(p)$$

is a maximal compact subgroup of G . Note that $K = \{I_{p+1,p+1}, I_{2p+2}\} \times K_0 \sim \mathbb{Z}_2 \times P$.

For $j = 1, \dots, p + 1$, let

$$H_j = \begin{pmatrix} 0 & X \\ X^t & 0 \end{pmatrix} \in \mathfrak{p}^\sharp \quad \text{where} \quad X = \mathrm{diag}(0, \dots, 0, 1, 0, \dots, 0),$$

with 1 on the j th position. Then $\mathfrak{t}^\sharp := \mathbb{R}H_1 + \cdots + \mathbb{R}H_{p+1}$ and $\mathfrak{t} := \mathbb{R}H_1 + \cdots + \mathbb{R}H_p$ are maximal abelian subspaces of \mathfrak{p}^\sharp and \mathfrak{p} . Let $\{\epsilon_j\}$ be the dual basis of $\{H_j\}$. The positive root systems of $(\mathfrak{g}^\sharp, \mathfrak{t}^\sharp)$ and $(\mathfrak{g}, \mathfrak{t})$ are $\{\epsilon_j \pm \epsilon_k, 1 \leq j < k \leq p+1\}$ and $\{\epsilon_j \pm \epsilon_k, 1 \leq j < k \leq p\}$.

2.2. The maximal parabolic subgroups P^\sharp and P .

We fix the elements:

$$\xi^\sharp = H_1 + \cdots + H_{p+1} \in \mathfrak{p}^\sharp \quad \text{and} \quad \xi = H_1 + \cdots + H_p \in \mathfrak{p}.$$

Let $\mathfrak{a}^\sharp = \mathbb{R}\xi^\sharp$, $\mathfrak{a} = \mathbb{R}\xi$. The root space decomposition of \mathfrak{g}^\sharp under ξ^\sharp and \mathfrak{g} under ξ is then

$$\mathfrak{g}^\sharp = \mathfrak{n}_{-2}^\sharp + \mathfrak{m}^\sharp + \mathfrak{a}^\sharp + \mathfrak{n}_2^\sharp, \quad \mathfrak{g} = \mathfrak{n}_{-2} + \mathfrak{n}_{-1} + \mathfrak{m} + \mathfrak{a} + \mathfrak{n}_1 + \mathfrak{n}_2,$$

with roots $\pm 2, 0$, and $\pm 2, \pm 1, 0$ respectively. We set for the corresponding positive root subspaces:

$$\mathfrak{n}^\sharp = \mathfrak{n}_2^\sharp \quad \text{and} \quad \mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2,$$

and for the negative root spaces:

$$\bar{\mathfrak{n}}^\sharp := \mathfrak{n}_{-2}^\sharp = (\mathfrak{n}_2^\sharp)^t \quad \text{and} \quad \bar{\mathfrak{n}} := \mathfrak{n}_{-1} \oplus \mathfrak{n}_{-2} = \mathfrak{n}^t.$$

We shall use explicit forms for the root spaces:

$$\mathfrak{n}^\sharp = \{n_Z^\sharp, Z \in \mathcal{X}_{p+1}\} \sim \mathcal{X}_{p+1} \quad \text{where} \quad n_Z := \begin{pmatrix} Z & -Z \\ Z & -Z \end{pmatrix},$$

and

$$\mathfrak{n} = \{\mathrm{diag}(n_{(z,v)}, 0); z \in \mathcal{X}_p, v \in \mathbb{R}^p\} \quad \text{where} \quad n_{(z,v)} := \begin{pmatrix} z & v & -z \\ -v^t & 0 & v^t \\ z & v & -z \end{pmatrix}.$$

We have:

$$\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \quad \text{with} \quad \mathfrak{n}_1 = n_{0, \mathbb{R}^p} \sim \mathbb{R}^p, \quad \mathfrak{n}_2 = n_{\mathcal{X}_p, 0} \sim \mathcal{X}_p.$$

The Lie algebra \mathfrak{n}^\sharp is abelian and \mathfrak{n} is a 2-step nilpotent Lie sub-algebra of \mathfrak{g} . Elements $n_{(z,v)}$ will simply be written as (z, v) . The Lie bracket in \mathfrak{n} is given, via the above identification $\mathfrak{n} = \mathcal{X}_p \oplus \mathbb{R}^p$, is

$$[z_1 + v_1, z_2 + v_2] = v_1 v_2^t - v_2 v_1^t.$$

Thus \mathfrak{n} is the free nilpotent Lie algebra with p generators (over \mathbb{R}).

The centralizer of \mathfrak{a}^\sharp in \mathfrak{g}^\sharp is

$$\mathfrak{m}^\sharp \oplus \mathfrak{a}^\sharp = \{l_{(X,Y)}^\sharp, X = X^t, Y^t = -Y \in \mathbb{M}_{p+1}\}, \quad l_{(X,Y)}^\sharp := \begin{pmatrix} Y & X \\ X & Y \end{pmatrix},$$

identified with $\mathfrak{gl}(p+1)$ via $l_{(X,Y)}^\sharp \mapsto X + Y \in \mathfrak{gl}(p+1)$, whereas the centralizer of \mathfrak{a} in \mathfrak{g} is

$$\mathfrak{m} \oplus \mathfrak{a} = \{\mathrm{diag}(l_{(x,y)}, 0); x = x^t, y^t = -y \in \mathbb{M}_p\}, \quad l_{(x,y)} := \begin{pmatrix} y & 0 & x \\ 0 & 0 & 0 \\ x & 0 & y \end{pmatrix},$$

identified with $\mathfrak{gl}(p)$. Note that

$$(2.1) \quad \mathfrak{gl}(p) \sim \mathfrak{m} \oplus \mathfrak{a} \subset \mathfrak{m}^\sharp \oplus \mathfrak{a}^\sharp \sim \mathfrak{gl}(p+1).$$

Let M^\sharp , A^\sharp , N^\sharp and \bar{N}^\sharp be the simply connected subgroup of G^\sharp with Lie algebras \mathfrak{m}^\sharp , \mathfrak{a}^\sharp , \mathfrak{n}^\sharp and $\bar{\mathfrak{n}}^\sharp$ respectively. Let $P^\sharp = M^\sharp A^\sharp N^\sharp$ and $\bar{P}^\sharp = M^\sharp A^\sharp \bar{N}^\sharp$ be the corresponding maximal parabolic subgroups of G^\sharp . Similarly we define the connected subgroups $P_0 = M_0 A N$ and $\bar{P}_0 = M_0 A \bar{N}$ of G_0 with Lie algebra $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ and $\mathfrak{m} + \mathfrak{a} + \bar{\mathfrak{n}}$. Note that the centralizer of \mathfrak{a}^\sharp in K^\sharp is

$$Z_{K^\sharp}(\mathfrak{a}^\sharp) = K^\sharp \cap M^\sharp = \{\text{diag}(k_1, k_1); k_1 \in \text{SO}(p+1)\} \sim \text{SO}(p+1),$$

and the centralizer of \mathfrak{a} in K is

$$Z_K(\mathfrak{a}) = K \cap M = \{\text{diag}(k_2, \det k_2, k_2, \det k_2); k_2 \in \text{O}(p)\} \sim \text{O}(p),$$

and the centralizer of \mathfrak{a} in K_0 is the connected component of the identity of $Z_K(\mathfrak{a})$. We set:

$$M = Z_K(\mathfrak{a})M_0, \quad P = Z_K(\mathfrak{a})P_0 = MAN \quad \text{and} \quad \bar{P} = Z_K(\mathfrak{a})\bar{P}_0 = MA\bar{N}.$$

The group $M^\sharp A^\sharp$ is isomorphic to the matrix group $\text{GL}_{p+1}^+ = \{h \in \text{M}_{p+1}, \det h > 0\}$ via:

$$(2.2) \quad h \in \text{GL}_{p+1}^+ \mapsto k_o^t \begin{pmatrix} h & 0 \\ 0 & h^{-t} \end{pmatrix} k_o \in M^\sharp A^\sharp, \quad k_o = 2^{-\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Restricting this isomorphism to $\{\text{diag}(h, \det h), h \in \text{GL}_p\}$ and to $\{\text{diag}(h, \det h), h \in \text{GL}_p^+\}$, we obtain an isomorphism between MA and GL_p and between M_0A and GL_p^+ . Thus, using the isomorphisms just above as identifications, P^\sharp , P and P_0 can be described as the semi-direct product:

$$P^\sharp = \text{GL}_{p+1}^+ N^\sharp, \quad P = \text{GL}_p N \quad \text{and} \quad P_0 = \text{GL}_p^+ N.$$

Lemma 2.1. *The following inclusions of Lie algebras hold:*

$$\mathfrak{m} \oplus \mathfrak{a} \subset \mathfrak{m}^\sharp \oplus \mathfrak{a}^\sharp \quad \text{and} \quad \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \subset \mathfrak{m}^\sharp \oplus \mathfrak{a}^\sharp \oplus \mathfrak{n}^\sharp.$$

We have:

$$M_0A \subset MA \subset M^\sharp A^\sharp \quad \text{and} \quad P_0 \subset P \subset P^\sharp.$$

Moreover we have a factorization of $\exp(n_{(z,v)}) \in N$ in $\text{GL}_{p+1}^+ = M^\sharp A^\sharp$,

$$(2.3) \quad \exp n_{(z,v)} = m \exp n_{M(z,v)}^\sharp,$$

where $m \in M^\sharp$ corresponds to $\begin{pmatrix} I_p & v \\ 0 & 1 \end{pmatrix} \in \text{SL}_{p+1}$ and

$$(2.4) \quad M(z,v) := \begin{bmatrix} z & \frac{1}{2}v \\ -\frac{1}{2}v^t & 0 \end{bmatrix} \in \mathcal{X}_{2q},$$

is viewed as an element of \mathfrak{n}^\sharp .

Proof. The first relation is in (2.1). We can write $n_{(z,v)} = l_{(X,Y)}^\sharp + n_W^\sharp$ with:

$$X = \begin{pmatrix} 0 & \frac{1}{2}v \\ \frac{1}{2}v^t & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & \frac{1}{2}v \\ -\frac{1}{2}v^t & 0 \end{pmatrix}, \quad W = \begin{pmatrix} z & \frac{1}{2}v \\ -\frac{1}{2}v^t & 0 \end{pmatrix}.$$

This shows $\mathfrak{n} \subset \mathfrak{m}^\sharp \oplus \mathfrak{n}^\sharp$ and implies $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \subset \mathfrak{m}^\sharp \oplus \mathfrak{a}^\sharp \oplus \mathfrak{n}^\sharp$. The group inclusions follow immediately. Easy matrix computations give the equality (2.3). \square

Note that the adjoint action of $MA = \text{GL}_p^+$ and GL_p on N is

$$(2.5) \quad g \cdot (z, v) = (gzg^t, gv).$$

3. DEGENERATE PRINCIPAL SERIES REPRESENTATIONS

Throughout the paper we assume p is odd and we write $2q = p + 1$.

3.1. Principal series of $G^\sharp = \mathrm{SO}_0(p+1, p+1)$.

Let $\mu \in \mathbb{C}$ and consider the induced representation $I^\sharp(\mu)$ of G^\sharp from the following character on P^\sharp

$$\chi_\mu^\sharp : me^{t\xi^\sharp} n \mapsto e^{(\mu+\rho^\sharp)t} ,$$

where $\rho^\sharp = q(2q-1)$ is the half sum of the positive root of $\mathrm{ad}\xi^\sharp$. In terms of $P^\sharp = \mathrm{GL}_{2q}^+ N^\sharp$ this is

$$(3.1) \quad \chi_\mu^\sharp : l n \mapsto \det(l)^{\frac{\mu+\rho^\sharp}{2q}} , \quad l \in \mathrm{GL}_{2q}^+ .$$

This representation can be realized on the space of Haar measurable functions $f(g)$ on G^\sharp such that

$$(3.2) \quad f(gln) = \det(l)^{-\frac{\mu+\rho^\sharp}{2q}} f(g), \quad l n \in P^\sharp ,$$

and

$$(3.3) \quad f|_{K^\sharp} \in L^2(K^\sharp) .$$

See [16]. The group G^\sharp acts on $I^\sharp(\mu)$ by the left regular action and we denote the representation by $(I^\sharp(\mu), \pi_\mu^\sharp)$. The condition (3.2) implies that $f \in I^\sharp(\mu)$ is invariant under the right action of $K^\sharp \cap M^\sharp$ and can therefore be identified as functions on $K^\sharp/K^\sharp \cap M^\sharp$. However $K^\sharp/K^\sharp \cap M^\sharp$ can be realized as $\mathrm{SO}(2q)$ since the group K^\sharp acts on $\mathrm{SO}(2q)$ by

$$K^\sharp \ni \mathrm{diag}(k_1, k_2) : \begin{cases} \mathrm{SO}(2q) & \longrightarrow \mathrm{SO}(2q) \\ a & \longmapsto k_1 a k_2^{-1} \end{cases} ,$$

and the isotropy group of the identity matrix $I_{2q} \in \mathrm{SO}(2q)$ is $K^\sharp \cap M^\sharp$. Thus condition (3.3) can be equivalently replaced by

$$f|_{K^\sharp/K^\sharp \cap M^\sharp} \in L^2(K^\sharp/K^\sharp \cap M^\sharp) = L^2(\mathrm{SO}(2q)) .$$

We denote by $(I_{K^\sharp}^\sharp(\mu), \pi_\mu^\sharp)$ the space of K^\sharp -finite elements. We will need its decomposition under K^\sharp . Recall, see e.g. [15], that each irreducible representation of $\mathrm{SO}(2q)$ is determined by a q -tuple of integers:

$$\underline{\mathbf{m}} = (m_1, \dots, m_q), \quad m_1 \geq \dots \geq m_{q-1} \geq |m_q| .$$

We write $\mathcal{V}_{\underline{\mathbf{m}}}$ for the representation space of $\underline{\mathbf{m}}$. Thus $I_{K^\sharp}^\sharp(\mu)$ is the same as the space $L^2(\mathrm{SO}(2q))_{K^\sharp}$ of K^\sharp -finite elements in $L^2(\mathrm{SO}(2q))$ and we have [15]:

$$(3.4) \quad I_{K^\sharp}^\sharp(\mu) \sim L^2(\mathrm{SO}(2q))_{K^\sharp} = \sum_{\underline{\mathbf{m}}} \mathcal{V}_{\underline{\mathbf{m}}} \otimes \mathcal{V}_{\underline{\mathbf{m}}}^* ,$$

where $\mathcal{V}_{\underline{\mathbf{m}}}^*$ stands for the dual representation of $\mathcal{V}_{\underline{\mathbf{m}}}$.

Each function in the space $I^\sharp(\mu)$ is also uniquely determined by its restriction to \bar{N}^\sharp . The G^\sharp -action in this realization is referred as \bar{N}^\sharp -realization. It is [16, p. 169] the space $L^2(\bar{N}^\sharp, e^{2\Re(\mu+\rho^\sharp)H^\sharp})$ where the function H^\sharp is defined by $t = H^\sharp(\bar{n})$ using the Iwasawa decomposition of $\bar{n} = kme^{t\xi^\sharp} n_+ \in K^\sharp P^\sharp$. Note however that the L^2 -norm in $L^2(K)$ or $L^2(\bar{N}^\sharp, e^{2\Re(\mu+\rho^\sharp)H^\sharp})$ is G^\sharp -invariant only for purely imaginary μ , $\mu \in i\mathbb{R}$.

3.2. Zeta distribution and complementary series $\mathcal{C}^\sharp(\nu)$ of G^\sharp .

The unitarity of $(I^\sharp(\mu), \pi_\mu)$ for μ outside the standard unitary range $\mu \in i\mathbb{R}$ has been completely determined in [15] in the K -finite realization; see also [20, 28]. Let $\mu \in (0, q)$. There exists a \mathfrak{g}^\sharp -invariant (i.e. \mathfrak{g}^\sharp acts as skew Hermitian operators) positive definite inner product $(\cdot, \cdot)_\nu$ on the space $I_{K^\sharp}^\sharp(\mu)$ of K^\sharp -finite elements of the principal series of $I^\sharp(\mu)$ (see [15, Theorem 7.5] and [2, section 7]). We will denote the unitary representation of G^\sharp on the completion of $I_{K^\sharp}^\sharp(\mu)$ as $(\mathcal{C}^\sharp(\mu), \pi_\mu^\sharp)$.

The non-compact picture has been further studied in [2] and we recall it briefly here. We identify \bar{N}^\sharp with \mathcal{X}_{2q} via $Z \mapsto \exp \bar{n}_Z^\sharp$ and we consider the following (formally defined) linear form on the Schwarz space $\mathcal{S}(\mathcal{X}_{2q})$

$$\mathbf{Z}_s(h) = \gamma_{s,2q} \int_{\mathcal{X}_{2q}} h(x) |\mathrm{Pf}(x)|^s dx, \quad \gamma_{s,2q} = \frac{\pi^{\frac{q}{2}(s+2q-1)}}{\prod_{j=0}^{q-1} \Gamma(\frac{s+2q-1}{2} - j)},$$

for $s \in \mathbb{C}$ with sufficiently large $\Re s$ where Pf denotes the Pfaffian polynomial. This defines [5, 2] a family of tempered distributions $\{\mathbf{Z}_s\}$ which admits a holomorphic continuation to the whole complex plane and whose Fourier transform satisfies the following functional equality:

$$(3.5) \quad \mathbf{Z}_{2q,s-(2q-1)}(\mathcal{F}h) = \mathbf{Z}_{2q,-s}(h), \quad s \in \mathbb{C};$$

here the Fourier transform on \mathcal{X}_{2q} is given by:

$$\mathcal{F}h(\zeta) = \int_{\mathcal{M}_{2q}^{ss}} h(x) e^{2\pi i(x,\zeta)} dx, \quad h \in \mathcal{S}(\mathcal{X}_{2q}),$$

The inner product in the N^\sharp -realization is given by:

$$(f, g)_\mu := (f, \mathbf{Z}_s * g)_{L^2(\mathcal{X})} = \mathbf{Z}_s(f * \check{g}) \quad \text{with} \quad s = \frac{\mu}{q} - (2q - 1),$$

where $\check{g} : x \mapsto \bar{f}(-x)$, initially defined on the Schwarz space $\mathcal{S}(\mathcal{X}_{2q})$. From (3.5), we see that $(f, f)_\mu = \mathbf{Z}_{-\frac{\mu}{q}}(|\mathcal{F}f|^2)$ for any $f \in \mathcal{S}(\mathcal{X}_{2q})$; this makes sense because for $\mu \in (0, q)$, $\mathbf{Z}_{-\frac{\mu}{q}}$ is a locally integrable function. The completion of $\mathcal{S}(\mathcal{X}_{2q})$ is

$$(3.6) \quad \mathcal{C}^\sharp(\mu) = \{f \in \mathcal{S}'(\mathcal{X}_{2q}); |\mathrm{Pf}|^{-\frac{\mu}{2q}} \mathcal{F}f \in L^2(\mathcal{X}_{2q})\},$$

(Note that the condition $f \in \mathcal{S}'(\mathcal{X}_{2q})$ can be deduced also from $|\mathrm{Pf}|^{-\frac{\mu}{2q}} \mathcal{F}f \in L^2(\mathcal{X}_{2q})$.)

We summarize the results in the following

Proposition 3.1. *Suppose $\mu \in (0, q)$. There exists a $(\mathfrak{g}^\sharp, \pi^\sharp)$ -invariant positive definite inner product $(\cdot, \cdot)_\mu$ on $I_{K^\sharp}^\sharp(\mu)$. Its completion defines a unitary representation of G^\sharp . In the non-compact realization the Hilbert space is described by (3.6).*

The operator

$$(3.7) \quad \mathcal{F}_\mu : f \in \mathcal{S}(\mathcal{X}_{2q}) \mapsto \phi = \mathcal{F}^{-1}(|\mathrm{Pf}|^{-\frac{\mu}{2q}} \mathcal{F}f) \in L^2(\mathcal{X}_{2q}),$$

then extends to a unitary operator from $\mathcal{C}^\sharp(\mu)$ onto $L^2(\mathcal{X}_{2q})$. We denote the corresponding representation on $L^2(\mathcal{X}_{2q})$ by

$$(3.8) \quad \tilde{\pi}_\mu^\sharp = \mathcal{F}_\mu \pi_\mu^\sharp \mathcal{F}_\mu^{-1}.$$

We obtain a simple description of the action of $(\bar{P}^\sharp, \tilde{\pi}_\mu^\sharp)$ (see [2, sections 7 and 8]):

$$(3.9) \quad \tilde{\pi}_\mu^\sharp(\exp \bar{n}_Z^\sharp) \phi(W) = \phi(W - Z)$$

and for $g = me^{t\xi^\sharp} \in M^\sharp A^\sharp$ identified with an element of GL_{p+1}^+ :

$$(3.10) \quad \tilde{\pi}_\mu^\sharp \phi(W) = e^{-\rho^\sharp t} \phi(g^t W g) .$$

3.3. Principal series $I(\nu)$ of G .

Let $I(\nu)$ be the principal series induced from the following character of $P = \mathrm{GL}_p N$:

$$\chi_\nu : l n \longmapsto |\det(l)|^{\frac{\nu+\rho}{p}} ,$$

where $\rho = p^2/2$ is the half sum of the positive roots of $\mathrm{ad}\xi$, with the similar condition as in (3.2) and (3.3). That is, the representation space is realized as Haar measurable functions $f(g)$ on G satisfying

$$(3.11) \quad f(g l n) = |\det(l)|^{-\frac{\nu+\rho}{p}} f(g) ,$$

and

$$(3.12) \quad f|_K \in L^2(K) \quad \text{or equivalently} \quad f|_K \in L^2(K/K \cap M) ;$$

the group G acts on $I(\nu)$ by the left regular action and we denote this action by $(I(\nu), \pi_\nu)$. We denote by $(I_K(\nu), \pi_\nu)$ the space of K -finite elements.

The homogeneous space $K/K \cap M$ can be realized as the Stiefel manifold of rank p isometries:

$$K/K \cap M = V_{p+1,p} = \{x \in M_{p+1,p}; x^t x = I_p\} ,$$

where the group K acts transitively via:

$$K \ni \mathrm{diag}(k_1, k_2) : \begin{cases} V_{p+1,p} & \longrightarrow V_{p+1,p} \\ x & \longmapsto k_1 x k_2^{-1} \end{cases} ,$$

and $K \cap M$ is the isotropy group of $\begin{pmatrix} I_p \\ 0 \end{pmatrix} \in V_{p+1,p}$. The elements f in $I(\nu)$ then satisfy

$$f|_{K/K \cap M} \in L^2(K/K \cap M) = L^2(V_{p+1,p}).$$

We will need the multiplicity free decomposition of $L^2(V_{p+1,p})$ under $K_0 = \mathrm{SO}(p+1) \times \mathrm{SO}(p)$. Let us recall that each irreducible representation of $\mathrm{SO}(p)$ is determined by a $(q-1)$ -tuples of integers

$$\underline{n} = (n_1, \dots, n_{q-1}), \quad n_1 \geq \dots \geq n_{q-1} \geq n_{q-1} \geq 0 ,$$

and we write $\mathcal{W}_{\underline{n}}$ for the representation space. Given a representation \underline{m} of $\mathrm{SO}(p+1)$ we write $\underline{m} \succeq \underline{n}$ if \underline{n} appears in the irreducible decomposition of \underline{m} under $\mathrm{SO}(p)$. It is a classical result, see e.g. [10, 26, 13] that \underline{n} appears in \underline{m} multiplicity free. This implies that the space of K_0 -finite elements of $L^2(V_{p+1,p})$ is decomposed under K_0 as follows:

$$(3.13) \quad I_K(\nu) \sim L^2(V_{p+1,p})_{K_0} = \sum_{(\underline{m}, \underline{n}) : \underline{m}^* \succeq \underline{n}} \mathcal{V}_{\underline{m}} \otimes \mathcal{W}_{\underline{n}} ,$$

and this decomposition is multiplicity free.

3.4. The restriction map R .

We shall consider simply the restriction of functions in $I^\sharp(\mu)$ to $G \subset G^\sharp$. To clarify its definition we note first that the space $I_{K^\sharp}^\sharp(\mu)$ of K^\sharp -finite functions are smooth functions on G^\sharp . Thus the restriction map

$$R : I_{K^\sharp}^\sharp(\mu) \mapsto C^\infty(G), \quad Rf(g) = f(g), \quad g \in G$$

makes sense. In the K^\sharp -realization of $I^\sharp(\mu)$, we have $Rf \in L^2(K)$ for any K^\sharp -finite elements $f \in L^2(\mathrm{SO}(2q))$.

Our main observation is the following:

Proposition 3.2. *Let $\nu, \mu \in \mathbb{C}$ such that $\nu = \frac{p}{p+1}\mu$. The restriction map R is a G -equivariant isomorphism from $I_{K^\sharp}^\sharp(\mu)$ onto $I_K(\nu)$ in the sense that*

$$R\pi_\mu^\sharp(g)f = \pi_\nu(g)Rf, \quad f \in I^\sharp(\mu), \quad g \in G,$$

and it is unitary as a map from $I_{K^\sharp}^\sharp(\mu) \sim L^2(\mathrm{SO}(2q))_{K^\sharp}$ onto $I_K(\nu) \sim L^2(V_{p+1,p})_{K_0}$.

Proof. Let $f \in I_{K^\sharp}^\sharp(\mu)$. By Lemma 2.1, one check easily for $ln \in P \subset P^\sharp$ $\chi_\mu^\sharp(ln) = \chi_\nu(ln)$. Together with (3.2), it implies that Rf satisfies (3.11). Moreover (3.3) implies (3.12). So $Rf \in I(\nu)$ and $R\pi_\mu^\sharp(g)f = \pi_\nu(g)Rf$ for any $g \in G$. As f is K^\sharp -finite, Rf is also K -finite.

The decompositions (3.13) and (3.4) show the rest of the claim. \square

Using Proposition 3.1 we get that restriction to G of the complementary series $\mathcal{C}^\sharp(\mu)$ defines a unitarizable representation of G , which we write as $\mathcal{C}(\nu)$, whose K -finite elements are the same as $I_K(\nu)$, by Proposition 3.2, i.e.,

$$\mathcal{C}(\nu) = R\mathcal{C}^\sharp(\mu), \quad \mathcal{C}_K(\nu) = I_K(\mu), \quad \nu = \frac{p}{p+1}\mu, \quad \mu \in (0, q).$$

The main result of this paper is the following theorem which states that restriction $\mathcal{C}(\nu)$ to the maximal parabolic subgroup \bar{P} of G is irreducible.

Theorem 3.3. *Let $\nu = \frac{p}{p+1}\mu$ with $\mu \in (0, q)$. Then restriction to G of the complementary series $(\mathcal{C}^\sharp(\mu), G^\sharp)$ defines a unitarizable irreducible representation $\mathcal{C}(\nu)$. It is the unitarization of the principal series representation $(I_K(\nu), G)$ realized in the non-compact picture. Moreover, it remains irreducible when restricted to the maximal parabolic subgroup $\mathrm{GL}_p \bar{N} = \bar{P}$.*

The irreducibility under G_0 in the above statement is essentially proved in [17]. Indeed let $\tilde{K} = \mathrm{Spin}(p+1) \times \mathrm{Spin}(p)$. The representation $\mathcal{C}(\nu)$ is treated as representation of $\mathrm{Spin}(p+1, p)$ and it is proved [17, 12.2.1] that the $(\mathfrak{g}, \tilde{K})$ -module of $\mathcal{C}(\nu)$ is irreducible. However the representations of \tilde{K} in $\mathcal{C}(\nu)$ descend to the representation of K_0 and thus the $(\mathfrak{g}, \tilde{K})$ -module is the same as (\mathfrak{g}, K) -module $\mathcal{C}_K(\nu)$, and the latter is then irreducible.

To prove the rest of Theorem 3.3, we will use the non-compact picture. As $\bar{P} \subset \bar{P}^\sharp$ we can find the action of \bar{P} on $(\tilde{\pi}_\mu^\sharp, L^2(\mathcal{X}_{2q}))$:

Lemma 3.4. *The representation of \bar{P} on $(\tilde{\pi}_\mu^\sharp, L^2(\mathcal{X}_{2q}))$ is unitarily equivalent to the representation $(\pi, L^2(N_p))$ given by:*

$$(3.14) \quad \pi(\bar{n}_0) \cdot \phi(\bar{n}) = \phi(\bar{n}_0^{-1}\bar{n})$$

for an element $n_0 \in \bar{N}$ and for an element $g \in \mathrm{GL}_p$:

$$(3.15) \quad \pi(g)\phi(\bar{n}) = |\det g|^{-\frac{p}{2}} \phi(g^{-1} \cdot \bar{n}) ,$$

where the action of GL_p on \bar{N} is given by (2.5).

Proof. Let us consider the unitary isomorphism:

$$\begin{array}{ccc} L^2(N_p) & \longrightarrow & L^2(\mathcal{X}_{2q}) \\ \psi & \longmapsto & \phi \end{array} \quad \text{given by} \quad \psi(z, v) = \phi(M(z, v)) .$$

It is easy to check (3.15). Now for $(z_o, v_o) \in N_p$, using (2.3), we compute:

$$\left(\exp \bar{n}_{M(z_o, v_o)}^\# m_h^\# \right) \cdot \phi(M(z, v)) = \phi(-M(z_o, v_o) + h^t M(z, v) h)$$

and direct computations show

$$-M(z_o, v_o) + h^t M(z, v) h = M((z_o, v_o)^{-1}(z, v))$$

so the action of \bar{N} is given by (3.14). \square

So by Schur's Lemma, Theorem 3.3 is proved once we have shown the following proposition:

Proposition 3.5. *Let T be a bounded operator on $L^2(\bar{N})$ commuting with the action π of \bar{P} defined in Lemma 3.4. Then T is the scalar multiple of the identity.*

Remark: Using the non-compact pictures, one can show the unitarity of $C(\nu)$. Indeed using Lemma 2.1 it is not difficult to show that the Knapp-Stein intertwiner $A_\mu^\#$ and $A_\nu^\#$ for the series $I^\#(\mu)$ and $I(\nu)$ satisfies:

$$(3.16) \quad A_\mu^\#(f^\#)(\exp \bar{n}_{M(z, v)}^\#) = A_\nu(f^\#|_G)(\exp \bar{n}_{(z, v)}) ,$$

and the properties of $A_\mu^\#$ described in [2] imply that I_ν is unitarizable.

Knapp-Stein intertwiners are (nilpotent) convolution operator with very singular kernels. By [2], A_μ is an abelian convolution with a power of the Pfaffian. It can be computed using geometric means that the kernel of A_ν is of the form $Q(z, v)^s = \det(z + \frac{1}{2}vv^t)^s$ for $(z, v) \in \bar{N}$ and it is only through some elementary but tricky matrix computations that it can be linked with the abelian convolution with some power of the Pfaffian as in (3.16).

4. PROOF OF PROPOSITION 3.5

We will use some well-known results for the Plancherel formula and von Neumann algebras of left regular representations of \bar{N} on $L^2(\bar{N})$; see [27, Chapt.14] and [7] for general locally compact groups [6] for the case of nilpotent groups. We describe first the support of the Plancherel measure described in [4, 8, 24].

4.1. The support of the Plancherel measure. To ease notation we write elements in $\bar{\mathfrak{n}}$ or \bar{N} as n instead of \bar{n} . We may identify them with elements of \mathcal{X}_{2q} by (2.4); on \mathcal{X}_{2q} we consider the standard inner product $(Z, W) = \frac{1}{2}\mathrm{Tr}ZW^*$. Hence we have equipped \mathfrak{n} with an inner product and we can now identify the dual \mathfrak{n}^* with \mathfrak{n} . The dual action of $g \in \mathrm{GL}_p$ on $\mathfrak{n}^* = \mathfrak{n}$ will be written as $g * n$.

We fix a generic point $o_{\mathfrak{n}^*} = (z_{o_{\mathfrak{n}^*}}, v_{o_{\mathfrak{n}^*}})$, the element of $\mathfrak{n}^* \sim \mathfrak{n}$ defined using (2.4) by:

$$M(o_{\mathfrak{n}^*}) = J_q \quad \text{where} \quad J_q = \mathrm{diag}(J, \dots, J) \in M_{2q} \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

It is easy to see that a representative of a coadjoint orbit can be chosen of the form $(z, v) \in \mathfrak{n}_p$ with $zv = 0$, that is, the vector v being in the kernel of the matrix z . Let \mathcal{O} be the collection of those representatives (z, v) with $M(z, v)$ non-singular:

$$\mathcal{O} := \{(z, v) \in \mathfrak{n}_p^*, zv = 0 \text{ and } \det M(z, v) \neq 0\}.$$

It is easy to see that \mathcal{O} is the following union of $SO(p)$ -orbits of certain "diagonal representatives":

$$\mathcal{O} := SO(p)\Delta * o_{\mathfrak{n}^*} \text{ where } \Delta = \{\text{diag}(d_1 I_2, \dots, d_{q-1} I_2, d_q), d_j \in \mathbb{R}^*\} \subset GL_p.$$

Any $w \in \mathcal{O} \subset \mathfrak{n}^*$ then induces an irreducible unitary representation λ_w of $\bar{\mathfrak{n}}$ and \bar{N} on $L^2(\mathbb{R}^{q-1})_w \sim L^2(\mathbb{R}^{q-1})$. We describe the representation of \mathfrak{n} very briefly for the element $o_{\mathfrak{n}^*} = (z_{o_{\mathfrak{n}^*}}, v_{o_{\mathfrak{n}^*}})$. The construction for general $w \in \mathcal{O}$ can be done similarly by using the equivariant action of GL_p on \mathfrak{n}^* . As the writing of $w \in \mathcal{O}$ as $GL_p \cdot o_{\mathfrak{n}^*}$ is not unique, there is a certain ambiguity here but it is harmless for our proof.

The element $o_{\mathfrak{n}^*}$ defines a splitting (or a complex structure) of $\mathbb{R}^{2(q-1)} = \mathbb{R}^{q-1} + \mathbb{R}^{q-1}$. The space \mathfrak{n} is decomposed as

$$(4.1) \quad \mathfrak{n} = \mathcal{X}_p \oplus \mathbb{R}^p = \mathfrak{n}_0 \oplus \mathfrak{h}$$

where $\mathfrak{n}_0 := (z_{o_{\mathfrak{n}^*}}^\perp \cap \mathcal{X}_p) \oplus \mathbb{R}v_{o_{\mathfrak{n}^*}}$ while $\mathfrak{h} := \mathbb{R}z_{o_{\mathfrak{n}^*}} + \mathbb{R}^{q-1} + \mathbb{R}^{q-1}$ is the Heisenberg algebra. There exists a unique representation $(\lambda_{o_{\mathfrak{n}^*}}, L^2(\mathbb{R}^{q-1}))$ of N whose restriction to $\exp \mathbb{R}o_{\mathfrak{n}^*}$ is given by the character $\exp i2\pi o_{\mathfrak{n}^*}$.

The Plancherel formula for $L^2(N)$ is given by

$$(4.2) \quad \|f\|_{L^2(\bar{N})}^2 = \int_{\mathcal{O}} \|\hat{f}(w)\|_2^2 d\iota(w), \quad f(0) = \int_{\mathcal{O}} \text{Tr}(\hat{f}(w)) d\iota(w),$$

where we have denoted the group Fourier transform of a function f by

$$\hat{f}(w) = \int_N f(g) \lambda_w(g) dg,$$

and its Hilbert-Schmidt norm by $\|\hat{f}(w)\|_2$ i.e. in $HS_w := L^2(\mathbb{R}_w^{q-1}) \otimes L^2(\mathbb{R}_w^{q-1})^*$. The Plancherel measure $d\iota$ can be explicitly computed but we will not need it here. In otherwords, the regular action of $N \times N$ on $L^2(N)$ is decomposed as

$$(4.3) \quad L^2(N) = \int_{\mathcal{O}} \lambda_w \otimes \lambda_w^* d\iota(w),$$

where λ_w^* is the contragradient of λ_2 .

4.2. Proof of Proposition 3.5. Let $\Omega := \{(z, v) \mathfrak{n}; \det(M(z, v)) \neq 0\}$. Clearly Ω is open and dense in \mathfrak{n} , and which strictly contained in $\Omega \subsetneq \mathcal{O}$. By elementary matrix computations, it can be also described as the orbit of $o_{\mathfrak{n}^*}$:

Lemma 4.1. *GL_p acts transitively on Ω and we have $\Omega = GL_p \cdot o_{\mathfrak{n}^*} = GL_p / \text{Sp}(q-1, \mathbb{R})$.*

Proof. Let $(z, v) \in \Omega$. The diagonalization of z provides a $g \in SO(p)$ such that $g \cdot (z, v) = (w, u)$ with $u = (u_1, \dots, u_p)$ and

$$w = \text{diag}\left(\begin{pmatrix} 0 & w_1 \\ -w_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & w_{q-1} \\ -w_{q-1} & 0 \end{pmatrix}\right).$$

We compute on the one hand $\det M(w, u) = (w_1 \cdots w_{q-1} u_p)^2$ and on the other hand $\det = \det M(g(z, v)) = \det g \det M(z, v) = \det M(z, v) \neq 0$. Namely $u_p, w_1, \dots, w_n \neq$

0. We solve now the equation $g(w, u) = o_{\mathfrak{n}^*}$ with $g \in \mathrm{GL}_p$ of the form $g = \begin{pmatrix} A & 0 \\ B & c \end{pmatrix}$ viz,

$$AZ_1 A^t = J_0 \quad , \quad AZ_1 B^t + AYc = 0, \quad cu_p = 1 \quad ,$$

which is easy to see to have a solution, e.g. by taking $c = v_p^{-1}$, $B = -v_p^{-1}Z_1^{-1}Y$ and

$$A = \mathrm{diag}(\mathrm{sgn} w_1 |w_1|^{\frac{1}{2}}, |w_1|^{\frac{1}{2}}, \dots, \mathrm{sgn} w_{q-1} |z_{q-1}|^{\frac{1}{2}}, |z_1|^{\frac{1}{2}}) \quad .$$

The isotropic subgroup in GL_p of $o_{\mathfrak{n}^*}$ consists of $g = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in \mathrm{GL}_p$ with $B = 0, C = 0, D = 1$ and A such that $AJ_{q-1}A^t = J_{q-1}$. Thus it is a realization of the symplectic group $\mathrm{Sp}(q-1, \mathbb{R})$ and we have $\Omega = \mathrm{GL}_p / \mathrm{Sp}(q-1, \mathbb{R})$. \square

There exists [9] a representation $(\tau, L^2(\mathbb{R}^{q-1}), \mathrm{Mp}(q-1, \mathbb{R}))$ of the double cover $\mathrm{Mp}(q-1, \mathbb{R})$ of $\mathrm{Sp}(q-1, \mathbb{R})$ such that

$$(4.4) \quad \tau(\tilde{g})\lambda_{o_{\mathfrak{n}^*}}(n)\tau(\tilde{g})^* = \lambda_{o_{\mathfrak{n}^*}}(g \cdot n), \quad \tilde{g} \in \mathrm{Mp}(q-1, \mathbb{R}) \quad ,$$

with $\tilde{g} \in \mathrm{Mp}(q-1, \mathbb{R}) \mapsto g \in \mathrm{Sp}(q-1, \mathbb{R})$ the double covering. Furthermore the representation $(\tau, \mathrm{Mp}(q-1, \mathbb{R}), L^2(\mathbb{R}^{q-1}))$ is a sum of two irreducible inequivalent representations [9, Theorem 4.56],

$$(4.5) \quad L^2(\mathbb{R}^{q-1}) = L_0^2(\mathbb{R}^{q-1}) \oplus L_1^2(\mathbb{R}^{q-1})$$

of even and odd functions.

We can now prove Proposition 3.5.

Proof of Proposition 3.5. Let T be a bounded operator on $L^2(N)$ commuting with the action π of \tilde{P} defined in Lemma 3.4.

As T commutes with the left translation, by the Plancherel Theorem [7], there exists a measurable field $\{\hat{T}(w), w \in \mathcal{O}\}$ of bounded operators on $L^2(\mathbb{R}_w^{q-1})$ such that

$$(4.6) \quad \widehat{Tf}(w) = \hat{f}(w)\hat{T}(w), \quad f \in \mathcal{S}(N) \quad ;$$

this measurable field of operator is unique (up to a ι -negligible set).

Let $w = g * o_{\mathfrak{n}^*}$ with $g \in \mathrm{SO}(p)\Delta$. By the orbit method, the representations λ_w and $n \mapsto \lambda_{o_{\mathfrak{n}^*}}(gn)$ are unitarily equivalent: there exists a unitary operator A_w such that $A_w \lambda_w(n) = \lambda_{o_{\mathfrak{n}^*}}(gn)A_w$ and, for $f \in \mathcal{S}(N)$, we compute with the change of variable $n_1 = gn$:

$$(4.7) \quad \begin{aligned} \hat{f}(w) &= \int_N f(n) A_w^{-1} \lambda_{o_{\mathfrak{n}^*}}(gn) A_w dn = A_w^{-1} \int_N (\pi(g)) f(n_1) \lambda_{o_{\mathfrak{n}^*}}(n_1) dn_1 A_w \\ &= A_w^{-1} \widehat{\pi(g)f}(o_{\mathfrak{n}^*}) A_w \quad . \end{aligned}$$

Now as T commutes with $\pi(g)$, we obtain easily:

$$\widehat{Tf}(w) = A_w^{-1} (T(\pi(g)f))^{\wedge}(o_{\mathfrak{n}^*}) A_w \quad .$$

Using (4.6) and the uniqueness of $\{\hat{T}(w), w \in \mathcal{O}\}$, we obtain for ι -almost all $w \in \mathcal{O}$,

$$(4.8) \quad \hat{T}(w) = A_w^{-1} \hat{T}(o_{\mathfrak{n}^*}) A_w \quad .$$

We may assume that $\hat{T}(o_{\mathfrak{n}^*})$ exists and that relation (4.8) holds for all $w \in \mathcal{O}$.

In the same way, we consider $\tilde{g} \in \text{Mp}(q-1, \mathbb{R})$ and the equivalence relation (4.4). Proceeding just as above, we obtain:

$$(4.9) \quad \hat{T}(o_{\mathfrak{n}^*}) = \tau(\tilde{g})\hat{T}(o_{\mathfrak{n}^*})\tau(\tilde{g})^{-1} .$$

It follows then from the irreducible decomposition (4.5) that $\hat{T}(o_{\mathfrak{n}^*})$ is constant on each space, namely

$$\hat{T}(o_{\mathfrak{n}^*}) = c_0 I + c_1 U$$

where U is the reflection,

$$Uh(x) = h(-x) \quad , \quad h \in L^2(\mathbb{R}^{q-1}) \quad , \quad x \in \mathbb{R}^{q-1} .$$

By the Plancherel Theorem [7], there exists a bounded operator $T_1 : L^2(N) \rightarrow L^2(N)$ which commutes with the left translations and satisfies

$$(4.10) \quad \widehat{T_1 f}(w) = \hat{f}(w)A_w^{-1}UA_w ,$$

for all $w \in \mathcal{O}$ and $f \in \mathcal{S}(N)$. Because of (4.8) and (4.9), we have $T = c_0 I + c_1 T_1$. So the proof of Proposition 3.5 will be over once we have shown that $c_1 = 0$ and for this it suffices to show that T_1 does not commute with the action of (π, GL_p) .

As T_1 is bounded on $L^2(N)$ and commutes with left translation, it is a convolution operator with a tempered kernel: there exists $\kappa \in \mathcal{S}'(N)$ such that $T_1 f = f * \kappa$ for any $f \in \mathcal{S}(N)$. We claim that κ is not invariant under GL_p and this shows that T_1 does not commute with the action of (π, GL_p) .

To show our claim, we first compute $T_1 f(0)$ for $f \in \mathcal{S}(\bar{N})$. For this we will use the Plancherel formula (see (4.2)):

$$(4.11) \quad T_1 f(0) = \int_{\mathcal{O}} \text{Tr}(\widehat{T_1 f}(w)) .$$

Now by (4.7) and (4.10), for $w = g * o_{\mathfrak{n}^*}$, we have:

$$(4.12) \quad \text{Tr}(\widehat{T_1 f}(w)) = \text{Tr}(\hat{f}(w)A^{-1}UA_w) = \text{Tr}(A_w \hat{f}(w)A^{-1}U) = \text{Tr}(\widehat{\pi(g)f(o_{\mathfrak{n}^*})}U) .$$

So we just want to compute the trace of $\hat{f}(o_{\mathfrak{n}^*})U$ on the Hilbert space $L^2(\mathbb{R}^{q-1})$. This can be derived from the standard formulas ([23, Chapt. XII, §6], [25, Chapt. II, §2-§3]) for the Weyl transform.

Indeed considering the decomposition (4.1), we write the elements of \mathfrak{n} as $h + h^\perp$ where $h \in \mathfrak{h}$ and $h^\perp \in \mathfrak{n}_0$. Integrating f over \mathfrak{n}_0 , we obtain the function F with \mathfrak{h} :

$$F(h) = \int_{\mathfrak{n}_0} f(h + h^\perp) dh^\perp .$$

We now identify \mathfrak{h} with the Heisenberg group and we write the elements of \mathfrak{h} as $h = (x, y, t) \in \mathbb{R}^{q-1} \times \mathbb{R}^{q-1} \times \mathbb{R}$. It is clear that $\hat{f}(o_{\mathfrak{n}^*})$ coincides with the Schrödinger representation of the Heisenberg group at F . From [23, Chapt. XII, §6.3], this shows that $\hat{f}(o_{\mathfrak{n}^*})$ is the integral operator on $L^2(\mathbb{R}^{q-1})$ with kernel

$$K_{\mathfrak{h}}(x, y) := c \int_{\mathbb{R}^{q-1} \times \mathbb{R}} e^{2\pi i(\frac{1}{2}u \cdot (y+x) + t)} F(u, y - x, t) du dt \quad , \quad x, y \in \mathbb{R}^{q-1} ,$$

where $c = c_q$ is a known constant (our t here corresponds to $\frac{1}{4}t$ in [23, Chapt. XII, §6.3, (91)]). So the kernel of the operator $\hat{f}(o_{\mathfrak{n}^*})U$ is $K_{\mathfrak{h}}(-x, y)$ and we can now

compute the trace of the operator:

$$\begin{aligned} \mathrm{Tr} \left(\widehat{f}(o_{\mathfrak{n}^*})U \right) &= \int_{\mathbb{R}^{q-1}} K_{\mathfrak{h}}(-x, x) dx \\ &= c \int_{\mathbb{R}^{q-1}} \int_{\mathbb{R}^{q-1} \times \mathbb{R}} e^{2\pi i t} F(u, 2x, t) du dt dx . \end{aligned}$$

Thus we have obtained:

$$\mathrm{Tr} \left(\widehat{f}(o_{\mathfrak{n}^*})U \right) = C(\mathcal{F}f)(o_{\mathfrak{n}^*}) ,$$

for some non-zero known constant C where we have denoted by \mathcal{F} the Euclidean Fourier transform on \mathfrak{n} , that is,

$$\mathcal{F}f(\zeta, \nu) = \int_{\mathfrak{n}} f(z, v) e^{2i\pi(\langle z, \zeta \rangle + \langle v, \nu \rangle)} dz dv ,$$

where we have used the canonical Euclidean scalar products on \mathcal{X}_{q-1} and \mathbb{R}^{q-1} .

Now by (4.12), this shows that for any $w \in \mathcal{O}$, we have:

$$\mathrm{Tr} \left(\widehat{f}(w)U \right) = C(\mathcal{F}f)(w) .$$

By (4.11), we obtain:

$$\int f(n) \kappa(n^{-1}) dn = T_1 f(0) = C \int_{\mathcal{O}} \mathcal{F}(w) d\iota(w) .$$

Hence the support of $\mathcal{F}\kappa(\cdot^{-1})$ is included in $\overline{\mathcal{O}}$. But $\overline{\mathcal{O}}$ is invariant under $n \mapsto n^{-1}$ but not invariant under GL_p since \mathcal{O} is strictly included in $\Omega = \mathrm{GL}_p \cdot o_{\mathfrak{n}^*}$ (see Lemma 4.1). So κ is not GL_p -invariant. This shows that T_1 does not commute with (π, GL_p) and concludes the proof of Proposition 3.5. \square

4.3. Decomposition of $L^2(N)$ under $\bar{N} \times \mathrm{Sp}(q-1, \mathbb{R})$. We note that the above proof also yields a decomposition of $L^2(N)$ under the action of $\bar{N} \times \mathrm{Sp}(q-1, \mathbb{R})$. Consider first the reference point $w = o_{\mathfrak{n}^*}$ and the space $L^2(\mathbb{R}^{q-1}) \otimes L^2(\mathbb{R}^{q-1})$ with N acting on the left factor by λ_w . Note that the metaplectic representation τ on $L^2(\mathbb{R}^{q-1})$, by its definition, defines a unitary representation of $\tau \otimes \tau^*$ of $\mathrm{Sp}(q-1, \mathbb{R})$ on $L^2(\mathbb{R}^{q-1}) \otimes L^2(\mathbb{R}^{q-1})$, viewed as Hilbert-Schmidt operators, by

$$(\tau \otimes \tau^*)(g)T = \tau(g)T\tau^*(g) .$$

The group $\bar{N} \times \mathrm{Sp}(q-1, \mathbb{R})$ acts on $L^2(\mathbb{R}^{q-1}) \otimes L^2(\mathbb{R}^{q-1})$ and we denote the corresponding representation, by $\lambda_w \rtimes \tau_w$.

Using the decomposition of $L^2(\mathbb{R}^{q-1})$ into even and odd functions $L^2(\mathbb{R}^{q-1})_i$, $i = 0, 1$, we have:

$$L^2(\mathbb{R}^{q-1}) \otimes L^2(\mathbb{R}^{q-1}) = L^2(\mathbb{R}^{q-1}) \otimes L^2(\mathbb{R}^{q-1})_0 \oplus L^2(\mathbb{R}^{q-1}) \otimes L^2(\mathbb{R}^{q-1})_1 ,$$

and we obtain the $\bar{N} \times \mathrm{Sp}(q-1, \mathbb{R})$ -irreducible decomposition:

$$\lambda_w \rtimes \tau_w = (\lambda_w \rtimes \tau_w)_0 + (\lambda_w \rtimes \tau_w)_1 .$$

Clearly this construction can be done for any ω . We have then

Corollary 4.2. *The space $L^2(N)$ is decomposed under $\bar{N} \times \mathrm{Sp}(q-1, \mathbb{R})$ as*

$$L^2(N) = \int_{\mathcal{O}} (\lambda_w \rtimes \tau_w)_0 + (\lambda_w \rtimes \tau_w)_1 d\iota(w) .$$

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